# In-Class Worksheet

CS 181 Advanced Algorithms — Fall 2025

**Hall's theorem:** Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then $G$ has a perfect matching if and only if $\forall\, S \subseteq L$ we have $|N(S)| \;\geq\; |S|$.

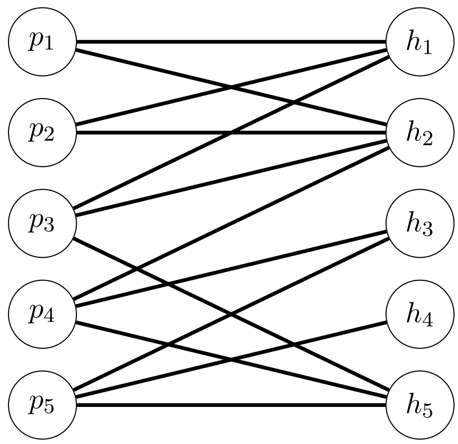


Figure 1: Determine the neighborhood $N(S)$ of $S = \{p_2, p_4\}$. Does this graph have a perfect matching? If yes, show it. If not, find a Hall-violating set.

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**Example 1**: A graph is $k$-regular if all nodes have degree $k$. Show every $k$-regular bipartite graph has a perfect matching.

**Example 2**: Consider a standard French deck of cards, with 4 suits and 13 values per suit, and shuffle it randomly. Deal 13 different piles, each pile containing 4 cards, the cards being face up. Show that you can always select exactly one card from each pile such that the 13 selected cards have the values A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K.

**Example 3**: Let $r, n \in \mathbb{N}$ with $r \leq n$. An $r \times n$ matrix $M$ is a *Latin rectangle* of size $(r, n)$ if $M_{ij} \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq r$, $1 \leq j \leq n$, and no number appears twice in any row or column. A *Latin square* is a Latin rectangle of size $(n, n)$.

For $r = 3$, $n = 5$, examples of Latin rectangles:

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

Non-examples:

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 1 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

(a) Show that two different Latin squares of size $3 \times 3$ exist.

(b) Let $M$ be a Latin rectangle of size $(r, n)$ with $r < n$. Show that one can add a row so the $(r+1, n)$ rectangle is also Latin. Repeatedly doing this shows that any Latin rectangle of size $(r, n)$ can be completed to an $n$ by $n$ Latin square.

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**Minimum-cost perfect-matchings:** Consider the bipartite graph $G = (L \cup R, E)$ with perfect matching $M \subseteq E$ shown in blue.

What is the current cost of $M$? Show that $M$ is not minimum cost by finding a net-negative alternating cycle. Toggle this cycle to obtain a new perfect matching $M'$ of smaller cost. Is $M'$ a minimum cost perfect matching?

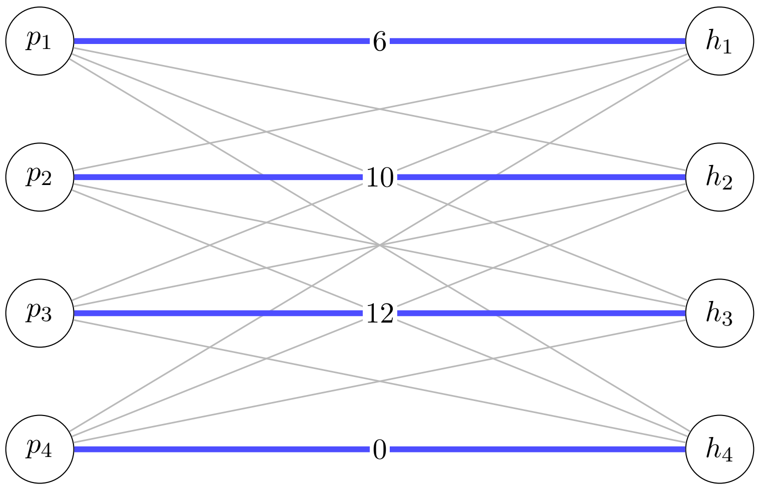


Figure 2: Unlabeled edges have cost 5.

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**Optimality Condition for Minimum Cost Bipartite Perfect Matching:** Let $G = (L \cup R, E)$ be bipartite graph with costs on edges and let $M \subseteq E$ be a perfect matching in $G$. Then $M$ is a minimum-cost perfect matching in $G$ if and only if: ________________________________.